Quantum mechanics. Department of Physics, 6th semester.

Lesson №2. ***Mathematical tools of quantum mechanics: operations with operators. Finding eigenfunctions and eigenvalues of Hermitian and non-Hermitian operators. Functions from operators.***

1. Operations with operators.

Tasks 1-6. Open the brackets in operator's expressions: $\left(\hat{A}+\hat{B}\right)^2$, $\left(\hat{A}-\hat{B}\right)^3$, $\hat{x}^2\frac{\hat{d}}{dx}$, $\frac{\hat{d}}{dx}\hat{x}^2$, $\hat{x}\frac{\hat{d}^2}{dx^2}$, $\frac{\hat{d}^2}{dx^2}\hat{x}$

Task 7. Calculate the commutator $\left[\hat{x}_i,\frac{\hat{\partial}}{\partial x_k}\right]$, where $i,k=1,2,3$ or x,y,z

Task 8. Calculate the commutator $[\hat{A},\hat{B}^n]$, if commutator $[\hat{A},\hat{B}]=1$.

Задача 9. Find out, which kind the operators listed below are (Hermitian, anti-Hermitian, or unitary):

$$\hat{L}\hat{L}^\dagger,\quad \hat{L}^\dagger\hat{L},\quad \hat{L}+\hat{L}^\dagger,\quad \hat{L}-\hat{L}^\dagger,\quad i\left(\hat{L}+\hat{L}^\dagger\right),\quad i\left(\hat{L}-\hat{L}^\dagger\right),$$

$$\hat{x},\quad \frac{\hat{d}}{dx},\quad i\frac{\hat{d}}{dx},\quad \frac{d^2}{dx^2},\quad \Delta,\quad \hat{x}\pm\frac{\hat{d}}{dx},\quad \hat{I},\quad \hat{T}_a,\quad \hat{P}_{12}$$

(**Def:** $\hat{L}^\dagger=\hat{L}$ - Hermitian operator, $\hat{L}^\dagger=-\hat{L}$ - anti-Hermitian operator,

$\hat{U}\hat{U}^\dagger=\hat{U}^\dagger\hat{U}=1$ - unitary operator)

2. Find eigenfunctions and eigenvalues of Hermitian and anti-Hermitian operators (*Hr. № 23, № 24, ГКК №1.34(a.b)*)

$$\frac{\hat{d}}{dx},\quad i\frac{\hat{d}}{dx},\quad \frac{d^2}{dx^2},\quad \hat{x}+\frac{\hat{d}}{dx},\quad \hat{x}-\frac{\hat{d}}{dx}$$

3. Function definition of the operator: $\hat{F}(\hat{f})=\sum_n C_n\hat{f}^n$, $\hat{F}(z)=\sum_n C_n z^n$.

Task 10. Find an explicit form of the operators:

1. $e^{i\varphi\hat{I}}$, $\hat{I}$ - operator of inversion; 2. $e^{a\frac{d}{dx}}$ (*НКК № 1.12*)

Task 11. Verify, that if $\hat{A}$ - Hermitian operator, then $e^{i\hat{A}}$ - unitary operator (*НКК № 1.59*)

Hometask: *Hr. №№ 11, 12, 23-29, 31; EK.1 №№ 3, 7, 8, 10, НКК №№1.13, 1.14.*

1. Find operators Hermitian conjugated to operators $\frac{\partial}{\partial x}$, $\frac{\partial^n}{\partial x^n}$ (Hr. № 11)

2. Finish *Hr. № 23, № 24, НКК №1.34(a,b).*

3. Find eigenfunctions and eigenvalues of operators $\frac{d}{d\varphi}$, $i\frac{d}{d\varphi}$, $0 \le \varphi < 2\pi$ (*Hr. № 25, № 27*)

4. Find eigenfunctions and eigenvalues of operators $\sin\left(\frac{d}{d\varphi}\right)$ $0 \le \varphi < 2\pi$ (*Hr. № 26*)

5. Find eigenfunctions and eigenvalues of operators $\exp\left(i\alpha\frac{d}{d\varphi}\right)$ $0 \le \varphi < 2\pi$ (*Hr. № 28*)

6. Find eigenfunctions and eigenvalues of operators $\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx}$ (*Hr. № 29*)

7. Find commutator of annihilation operator $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{x} + i\hat{p}\right)$ and birth operator $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{x} - i\hat{p}\right)$. (*Hr. № 31*)

8. May $\hat{A}_1 = \frac{1}{4}\left(\hat{a}^2 + (\hat{a}^\dagger)^2\right)$, $\hat{A}_2 = \frac{1}{4}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$; $\hat{A}_2 = \frac{i}{4}\left(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right)$; $\left[\hat{a}, \hat{a}^\dagger\right] = 1.$

Calculate commutators $\left[\hat{A}_i, \hat{A}_k\right], i, k = 1, 2, 3.$ (*ЕК Hr. 1, № 7*)

9. May $\hat{C}_1 = \frac{1}{2}\left(\hat{c} + \hat{c}^\dagger\right)$; $\hat{C}_2 = \frac{1}{2}\left(\hat{c}^\dagger - \hat{c}\right)$; $\hat{C}_3 = \frac{1}{2}\left(\hat{c}^\dagger\hat{c} - \hat{c}\hat{c}^\dagger\right)$; $\hat{c}^\dagger\hat{c} + \hat{c}\hat{c}^\dagger = 1; \hat{c}^2 = 0.$

Calculate commutators $\left[\hat{C}_i, \hat{C}_k\right], i, k = 1, 2, 3.$ (*ЕК Hr. 1, № 10*)

10. Supposing λ is the small quantity , find expansion of operators $\left(\hat{A} - \lambda\hat{B}\right)^{-1}$ in powers of λ. (*НКК №№1.13*)

11. Prove formula: $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \dots$

(*НКК №№1.14*)

12. Prove formula: if $\left[\hat{b}, \hat{a}\right] = i\lambda$, then

$$\exp\left[x\left(\hat{a} + \hat{b}\right)\right] = \exp\left(x\hat{b}\right)\exp\left(x\hat{a}\right)\exp\left(-\frac{i\lambda x^2}{2}\right).$$

EK – Elyutin P.V., Krivchenkov V.D. Quantum mechanics 1976
HKK- Halitskii E.M., Karnakov B.M., Kohan V.I. Problems in Quantum Physics, 1981
Hr. - Hrechko, Suhakov, Tomasevich, Fedorchenko Collection of theoretical physics problems, 1984